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Scaling exponents for random walks on Sierpinski carpets and number of distinct sites visited: a new algorithm for infinite fractal lattices

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Abstract. The scaling exponent for the mean square distance covered in a random walk (d_w) and the average number of distinct sites visited (d_n) are determined for a family of Sierpinski carpet patterns. We suggest a new random walk algorithm to generate walks on an effectively infinite deterministic fractal lattice. The algorithm is applied to several Sierpinski carpet patterns with the same Hausdorff dimension. We show that the systems have a quite different scaling exponent d_w and, further, that the generally accepted result $d_n = d_s$ does not hold for all of these, where d_s is the spectral dimension.

1. Introduction

Several features of the random walks (RWs) on fractal lattices are being extensively investigated at present [1–3]. It is now well known that diffusion on a fractal is anomalous, with the mean square distance travelled in time t given by

$$\langle r^2 \rangle \sim t^{2/d_w}. \tag{1}$$

The walk exponent is usually greater than two for fractals. For a RW on a fractal lattice, t is equivalent to the number of steps N.

Identification of the factors which determine d_w is still a very elusive problem. Some of the features which may be important are the following.

Ramification. Ramification of the pattern means branching. Actually, the fractal dimension d_f provides a very broad indication of the structure and fractals with the same d_f can have very different patterns and therefore properties. Ramification specifies some further information. Fractal patterns are classified as 'finitely' and 'infinitely' ramified, according to the number of points which are to be disconnected in order to isolate a portion of the pattern [1, 4]. If a finite number of cuts isolate a certain region, the fractal is said to be finitely ramified, e.g. the Sierpinski gasket pattern. But in the case of Sierpinski carpets, to cut out a section one has to cut out line segments rather than points and thus ramification is infinite. Another feature is the presence or absence of loops in the structure. In a loopless fractal any two arbitrary points are connected by only one path.

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Chemical Distance. The minimum path length *along the structure* between two arbitrary sites on a fractal is called the chemical distance, l. The average square distance from the origin on the fractal scales with l as

$$\langle r^2 \rangle \sim l^{2d_l}.$$

d_l is the chemical distance exponent.

While for the simpler class of finitely ramified loopless fractals d_w can be exactly determined [1], there is no such prescription for structures with loops. For looped structures which are finitely ramified it is possible to determine d_w from a scaling of the transit times, but no exact relationship of d_w with the the fractal dimension d_f and chemical distance exponent d_l exists. Such a procedure has been applied to the Sierpinski gasket [1, 5].

For infinitely ramified fractals the transit time scaling method is also inappropriate and so the only possible way to determine d_w is through RW simulations.

The principle objectives of this work are as follows:

(1) To study the variation of d_w in different fractal patterns with the same d_f, and to try to identify the features of the pattern which determine d_w.

(2) The other problem we address is the variation of the average number of distinct sites visited on the fractal within the duration of the walk.

It is found that this number varies with t (or N) according to the power law

$$S_N \sim t^{d_n/2}. \tag{2}$$

This d_n is usually considered equal to the spectral dimension d_s obtained from the scaling of the density of states of vibration on the fractal with frequency [6, 7]

$$\rho(\omega) \sim \omega^{d_s - 1}. \tag{3}$$

However, there appear to be exceptions to this rule [8–10]. The equivalence of d_n and d_s is not rigorously established and the conditions under which they may become distinct is not clearly specified.

In a preliminary study on Sierpinski carpets we showed that d_n and d_s may not always be equal [10]. However, that work suffered from inaccuracies due to finite-size effects.

In this paper we develop and use a new RW algorithm to study Sierpinski carpets with larger scaling factors, of infinite as well as finite ramification, with and without loops. We have also repeated the calculations on the earlier patterns studied.

The salient feature of the new RW method is that the walker sees an effectively infinite lattice so our results are much more reliable. The problems arising due to the boundary in the finite system are eliminated. The only limitation on the walk is due to the availability of computer time.

The question of identifying the features of the pattern affecting d_w is still far from answered, but our results indicate that the presence of loops on all sizes is necessary for d_n and d_s to be equivalent.

The organization of the paper is as follows. We discuss in section 1.1 the problem of the determination of d_w in further detail and also the earlier works on this subject; in section 1.2 we discuss work on the number of distinct sites visited in the RW. Section 2 describes the carpet patterns we have studied. In section 3 our new RW algorithm is presented. In section 4, we give the results obtained and discuss implications of the results.

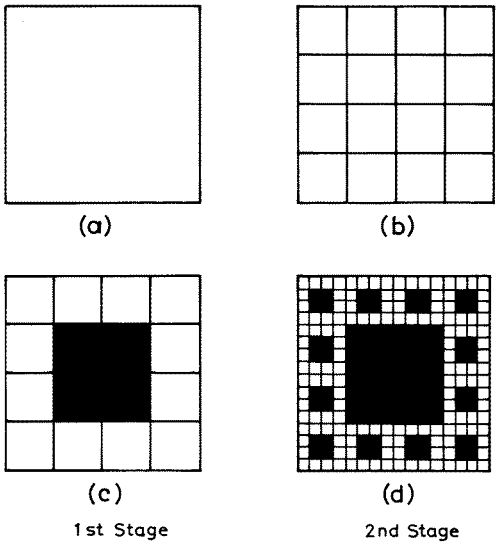

Figure 1. Construction of a Sierpinski carpet with $b = 4$ and $m = 12$. (a)–(c) The first stage, and (d) the second stage. Shaded squares are blocked.

1.1. *Diffusion in carpets*

The structure of the two-dimensional Sierpinski carpet pattern is as follows: a square of size say 'L' is divided into $b \times b$ equal smaller squares, as shown in figure 1. 'm' of the smaller squares are occupied and the remaining $(b^2 - m)$ vacant. This gives the generator, or first stage pattern. Higher stages are obtained by repeating the division process on each of the occupied squares. Continuing this indefinitely gives a fractal pattern with Hausdorff dimension

$$d_f = (\log m / \log b).$$

Now by arranging the occupied squares differently various patterns can be created each having the same Hausdorff dimension d_f. The Sierpinski carpets may have finite or infinite ramification as we shall show later.

A RW along the occupied sites will give the walk exponent from equation (1). The value of d_w is determined not only by d_f but also by other details of the generator. It is our aim to try to identify these features.

As discussed in the previous section, no exact prescription has been found to determine d_w when the generator is an infinitely ramified and looped structure, but attempts have been made to frame some approximate rules.

Such an approximate calculation has been performed by Gefen *et al* [11] on Sierpinski carpets by a bond-moving renormalization group (RG) approach on a resistance network of connected bonds. They obtained the resistance exponent and calculated the diffusion exponent from it, using the fractal Einstein relation [1, 2].

Another calculation on Sierpinski carpets considering bulk resistance was suggested by Kim *et al* [12], showing bounds of d_w. They showed that their simulation results lie within these bounds. Their work considers only symmetric carpets, moreover, finer details of the pattern are not considered.

Several attempts have been made to identify a precise geometric characteristic which will give the variation in d_w for structures having the same d_f. Some have suggested 'lacunarity' as the appropriate quantity [12–16]. This term was first introduced by Mandelbrot [4] as a feature distinguishing between patterns of the same d_f but with different arrangements. Lacunarity, as the name implies, is a measure of the size distribution of the 'lacunae' or holes in the pattern. It is also related to the degree of departure from translational invariance. However, there is still no very general definition of lacunarity which is applicable to all fractals, so it is not such a useful approach.

In a preliminary study reported by our group [10] the diffusion behaviour for a number of carpet patterns with the same d_f but a different generator were reported. But these results are rather inaccurate due to finite-size effects. This work is an extension of the study, where we repeat the earlier work on $b = 4$ patterns with greater accuracy and also study larger and more varied carpet patterns with $b = 7$.

The variation in d_w obtained from our observations has been correlated with the average number of nearest neighbours per site A_{nn} and the area–perimeter ratio β. These two quantities give an estimate of the compactness of the structure. We find, besides, that for infinitely ramified carpets, the distribution of loop sizes plays a significant role.

1.2. Distinct sites visited

During a RW a walker may visit a certain site a number of times, unless the algorithm specifies a self-avoiding walk. A quantity of considerable interest in such cases is the average number of distinct sites S_N visited in a walk of N steps. Here a site visited repeatedly is counted only once. Obviously $S_N \leqslant N$, where the equality holds for a self-avoiding walk.

The quantity S_N is important in the study of diffusion-controlled reactions and the kinetics of trapping processes [1].

The exponent d_n controlling S_N and the spectral dimension d_s are usually assumed equal. Let us examine the logic behind this premise. From the equivalence of the vibration and diffusion problem, Alexander and Orbach derived the relation [7]

$$d_f/d_w = d_s/2. \tag{4}$$

In the limiting case of a Euclidean lattice this relation remains valid, since in this case $d_w = 2$ for all dimensions and

$$d_f = d_s = d$$

is the Euclidean dimension. The exponent d_n is introduced in the following manner. It is assumed that the number of distinct sites visited in N steps is proportional to the volume accessible in N steps:

$$S_N = V(N). \tag{5}$$

This leads to

$$S_N = N^{d_f/d_w}$$

and hence

$$d_n = d_s.$$

The crucial assumption in this derivation is equation (5), which may not be valid in general where $d_s \leqslant 2$ is not satisfied. The simplest case in which it is violated is the Euclidean system for dimension >2. Here the diffusion law is

$$\langle r^2 \rangle \sim N$$

and the distinct sites visited scale as follows [17]:

$$S_N \sim N^{1/2}$$

for one dimension,

$$S_N \sim N/\log N$$

for two dimensions and

$$S_N \sim N$$

for three and higher dimensions.

So d_n and d_s are not equivalent for dimensions higher than two. Obviously, d_n cannot be greater than two since S_N cannot increase faster than N (or t).

In higher dimensions each site has a larger number of nearest neighbours and the probability of repeatedly visiting the same sites diminishes. $V(N)$ at fixed N is, however, an increasing function of dimension.

It is asserted [6] that equation (5) is true for $d < 2$ for Euclidean space and by analogy true for fractals with $d_s \leqslant 2$. The reasoning behind this is not very clear. Havlin *et al* [18] have studied RW with a large number of walkers for systems with $d_s < 2$. They find the N (or t) dependence of the number of sites visited as in equation (5). Others have suggested fractal systems where this relation does not hold. Dhar and Ramaswamy [8] and Nakanishi and Hermann [19] give the example of tree-like or comb-like structures. Here the random walker may get trapped for a long time on a single branch and thus cannot sample the whole fractal system uniformly. In this case equation (5) does not hold: in structures such as this, the exploration is not 'compact'. They argue that the presence of loops lets the walker wander more freely in the whole structure and reduces the difference between d_n and d_s.

We have calculated the two exponents, d_n and d_s, separately for various Sierpinski carpet patterns with the same fractal dimension. d_n has been calculated directly, using the mean S_N obtained from RW simulations. On the other hand, d_s is calculated through relation (4) using d_w, obtained from RW simulations. A significant structure-dependent deviation

$$\delta = d_n/2 - d_s/2 \tag{6}$$

is found; this is discussed in later sections.

Our work on Sierpinski carpets of different patterns illustrates cases where δ is negligibly small, as well as cases where it has a nonzero value outside the limits of error.

We find that for carpet blocks which consist of compact regions of accessible sites separated by compact blocks of inaccessible sites, δ is nonzero even when loops are present. For finitely ramified patterns δ increases with A_{nn}. This is presumably because such patterns are closer to a Euclidean lattice. On the other hand, patterns which are loopless also have a large δ as suggested in [8, 19]. It appears that for δ to approach zero the presence of loops on all sizes is required, i.e. there must be a broad and uniform distribution of loop sizes. Unfortunately, the error in our estimation of δ is so large that we cannot attempt to correlate the magnitude of δ with any definite feature of the pattern.

It is indeed diffficult to draw any final conclusion on such a complex problem from the study of such a small number of patterns. Much work is still necessary to find the appropriate features of the pattern which determine δ.

2. The carpet patterns

Our carpet pattern generators consist of a mesh of $b \times b$ small squares, of which 'm' are occupied. The generator is repeated self-similarly on the occupied squares to give the fractal pattern. We have studied one smaller set of patterns with $b = 4$ and $m = 10$, and another set with $b = 7$ and $m = 30$. The generators are shown in figures 2–4. Second-stage patterns of

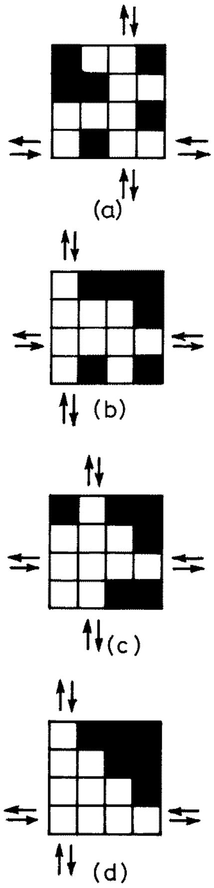

Figure 2. Generators of the 4×4 carpets studied. Arrows indicate positions of the connected paths which will appear in the second stage. Shaded squares are blocked.

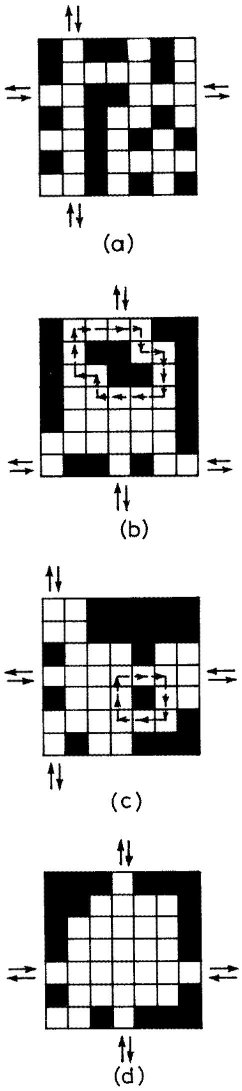

Figure 3. Generators for the finitely ramified 7×7 carpet patterns studied. Shaded squares are blocked.

a few selected carpets are shown in figures 5 and 6. All patterns belonging to the the first set have the same fractal dimension

$$d_f = \log 10/\log 4 = 1.66$$

and the second set

$$d_f = \log 30/\log 7 = 1.748.$$

We have tried to make the patterns as different from each other as possible, keeping only the total number of occupied squares constant, for each set. The only restriction imposed is that the accessible sites are all connected to each other, leaving no isolated open regions. This is neccessary to prevent the random walker from becoming trapped.

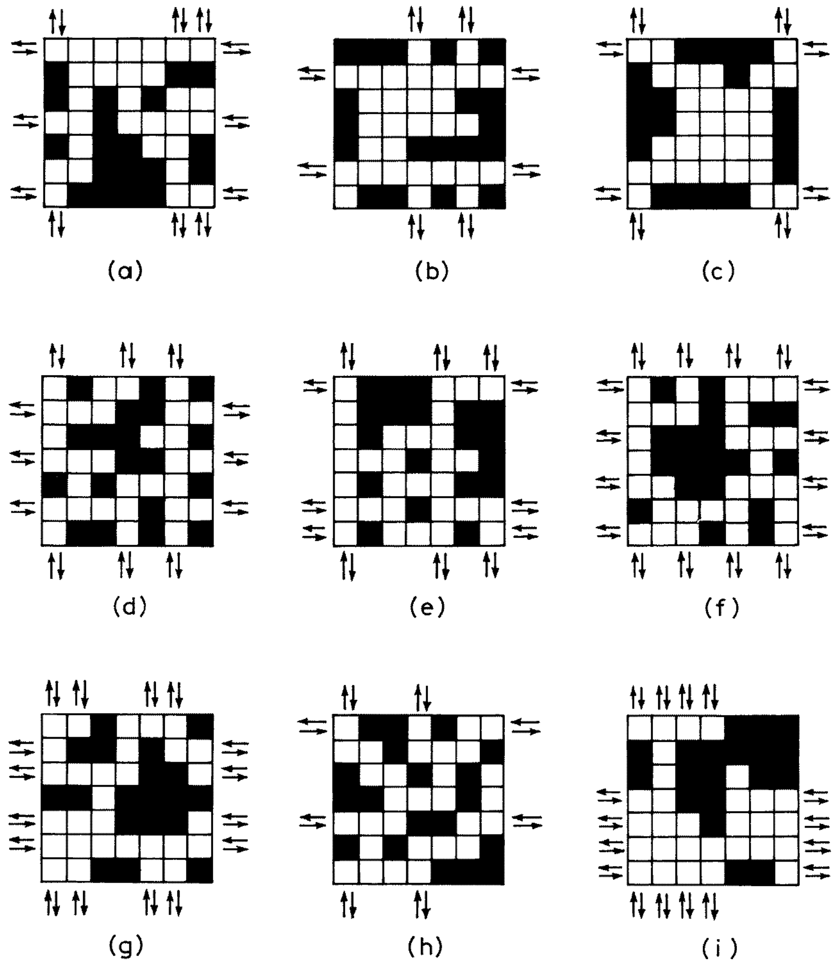

Figure 4. Generators of the infinitely ramified carpet patterns studied. Multiple entry points shown by the arrows indicate infinite ramification. Shaded squares are blocked.

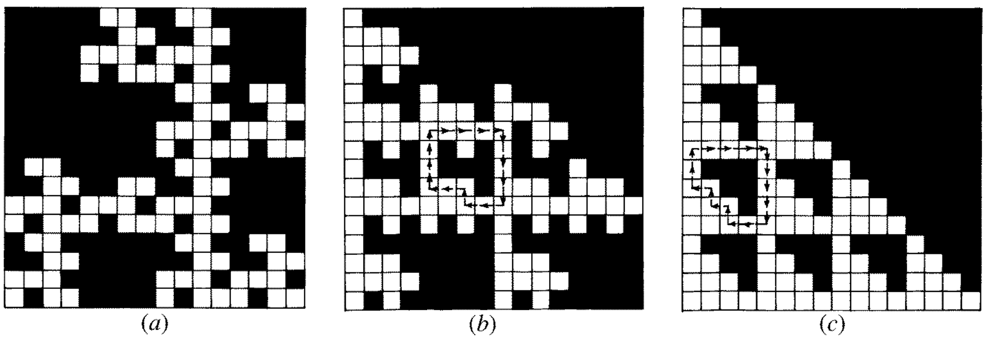

Figure 5. (*a*)–(*c*) represent, respectively, the second stages of the motifs in figures 2(*a*), (*b*) and (*d*). The shaded squares are blocked.

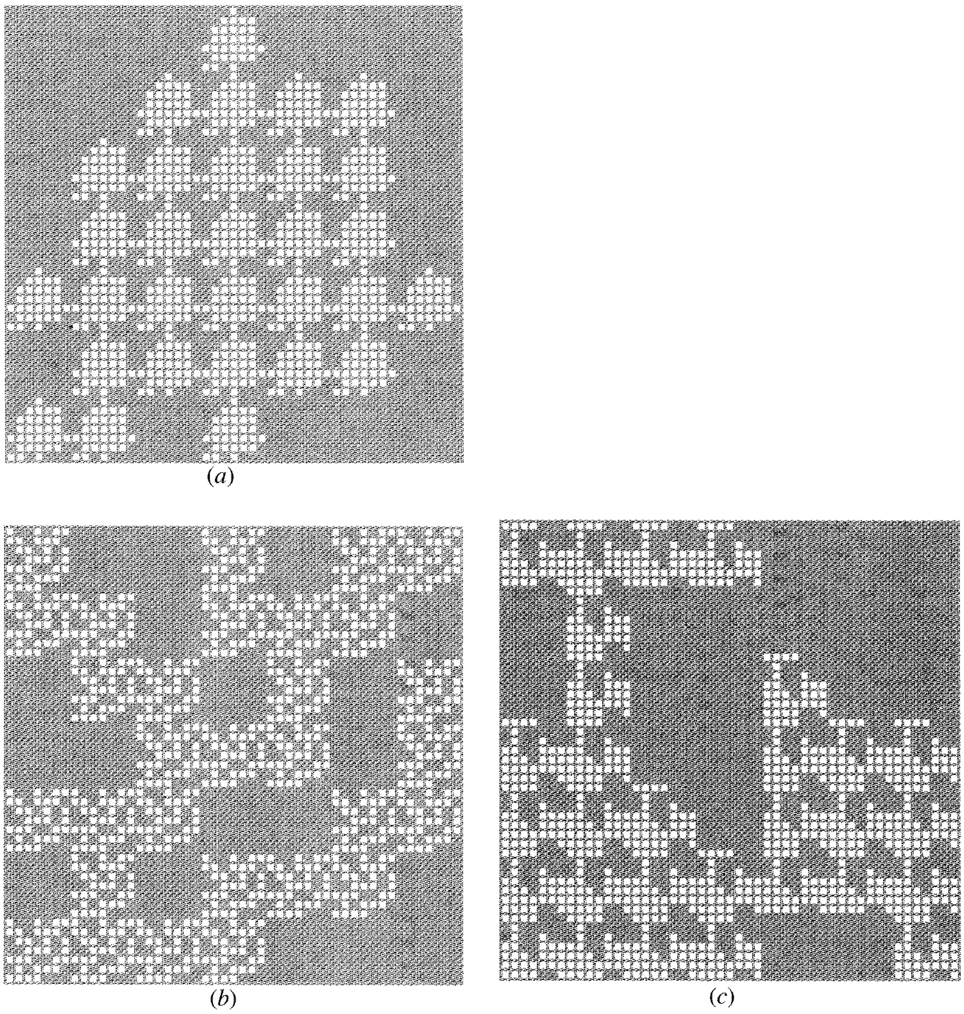

Figure 6. (*a*)–(*c*) show the second stages of 3(*d*), 4(*h*) and 4(*i*), respectively. The shaded region is inaccessible.

When the generators are arranged to form the higher-stage pattern, the number of entry points between adjacent lower-stage motifs determines the ramification. For patterns where there is only one entry point connecting opposite sides of the squares as those shown in figures 2 and 3 the ramification is finite. But when multiple entry points are possible the pattern is infinitely ramified, as in figure 4. The entry points where opposite sides match are indicated by double arrows in the figures.

To obtain a quantitative measure of the 'difference' between the patterns of the same set, we have used two quantities. One is the average number of nearest neighbours per site A_{nn}, and the other is the area–perimeter ratio β. These two measures are found to be almost equivalent. Some properties of the RW appear to be correlated to a systematic variation of either of these two parameters. However, they alone are not sufficient, especially in the case of the patterns which are infinitely ramified.

The problem of characterization is not fully solved yet, we propose as further significant features the presence of 'clusters' and 'loops' in the pattern. These terms are defined later. But we do not yet have a satisfactory quantification for these two features.

The patterns with $b = 4$ in our study are all finitely ramified, but the $b = 7$ motifs have infinite ramification in some cases. For the patterns which are loopless as well as being finitely ramified d_w can be calculated using the relation given by Havlin and Avraham [1], if d_l is known. For such patterns our simulated results agree very well with the calculated exponents. This gives us further confidence in the RW procedure used.

In analysing the patterns we use the term 'loop' where one or more blocked sites are surrounded by accessible sites. The size of the loop formed is measured by the smallest possible closed path enclosing the blocked sites. By the term 'cluster' we mean a compact group of accessible sites. The size of a cluster is measured by the number of such adjacent sites. We require a 'cluster' to be formed by at least four adjacent sites, two along the x-axis and two along the y-axis. It is to be noted that a loop or cluster may not be apparent in the generator of the pattern, but may be formed in the second or higher stages.

Having formed the fractal carpet pattern we proceed with the RW along accessible sites, according to the algorithm described in the next section.

3. The RW Algorithm

In our earlier work [10] we generated carpet patterns up to the fifth stage, and performed RWs on them using the blind-ant algorithm [1]. Walks which reached the boundary of the fifth-stage pattern were terminated and not taken into account. Though the number of walks thus discarded was very small, the process affected the average results considerably. This is because the walks terminated were the longest ones.

In our new algorithm, we do not store a generated pattern up to some definite stage. We only supply the generator as input. The random walker is set on some site and it tests whether each site it arrives at is an allowed site, as it goes along. The procedure is as follows.

The walk is confined to the X-positive and Y-positive quadrant only, in the two-dimensional plane. The initial site is chosen to have coordinates larger than the total number of steps to be travelled. To check whether a site is accessible the first step is to identify the stage the point belongs to. For a $b \times b$ carpet, a point having either an X- or Y-coordinate between b^{n-1} and b^n belongs to the nth stage. In the nth-stage coarse-grained pattern with units of size b^{n-1}, it is checked whether the block containing the site matches an accessible site on the given motif. If found accessible, the corresponding point in the next lower stage, i.e. $(n - 1)$, is ascertained. In this way the point is successively scaled down until it reaches the first stage. Finally, if the position in the first stage matches an accessible point then it is an allowed site. If the point corresponds to a blocked site, at any stage of the process, it is inaccessible. We illustrate the algorithm more fully with examples in the appendix. Further details are given in [20].

The advantage is that the system becomes effectively infinite without requiring storage of a large number of points. The self-similarity of the fractal lattice is fully utilized here. Algorithms have been proposed previously for RWs on an effectively infinite lattice, e.g. on percolation clusters [21]. But there, although the boundary effect is absent, a record of previously visited sites must be maintained since the disorder is quenched, i.e. a site once blocked must always be inaccessible.

In the present case, the pattern being deterministic, the problem mentioned above does not arise and each site is separately checked for accessibility as the walk proceeds.

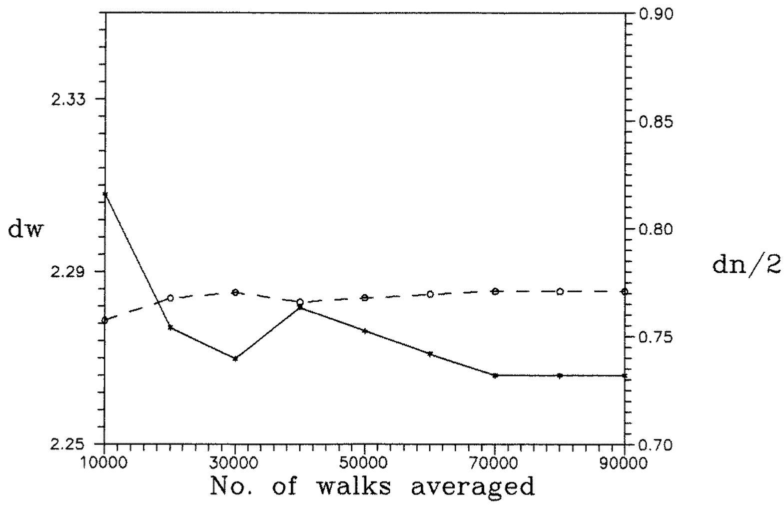

Figure 7. Variation in d_w and $d_n/2$ with the number of walks averaged over for the pattern of figure 4(*e*).

Table 1. Results for $b = 7$ carpet patterns, i and f denote infinite and finite ramification.

Carpet	Ram	A_{nn}	β	$d_w \pm 0.002$	$\delta \pm 0.01$
4(*a*)	i	2.86	0.88	2.194	0.00
4(*b*)	i	2.52	0.68	2.255	0.00
4(*c*)	i	2.37	0.61	2.266	0.02
4(*d*)	i	2.64	0.74	2.289	0.00
4(*e*)	i	2.27	0.59	2.360	0.03
4(*f*)	i	2.07	0.54	2.388	0.02
4(*g*)	i	2.54	0.68	2.419	0.04
4(*h*)	i	2.07	0.52	2.547	0.03
4(*i*)	i	3.00	0.99	2.495	0.07
3(*a*)	f	3.03	0.94	2.571	0.09
3(*b*)	f	2.76	0.81	2.645	0.08
3(*c*)	f	2.69	0.76	2.675	0.07
3(*d*)	f	2.00	0.50	2.746	0.05

4. Results

In this section we present the results for the observed variation in d_w and the approximate value obtained for δ.

RWs on complicated fractal lattices require averaging over a large number of walks to give converging results. We began each walk from a different point chosen randomly. We found that averaging over at least 70 000 such walks gives a value of d_w and d_n converging to within 0.002. In figure 7, we show how these exponents converge on increasing the number of walks. The walks are of 2 000–10 000 steps.

4.1. Variation in d_w

First we discuss the results for the $b = 7$ carpets, as it was possible to produce a greater variety of these patterns. The values for d_w obtained from the simulations are given in table 1. There

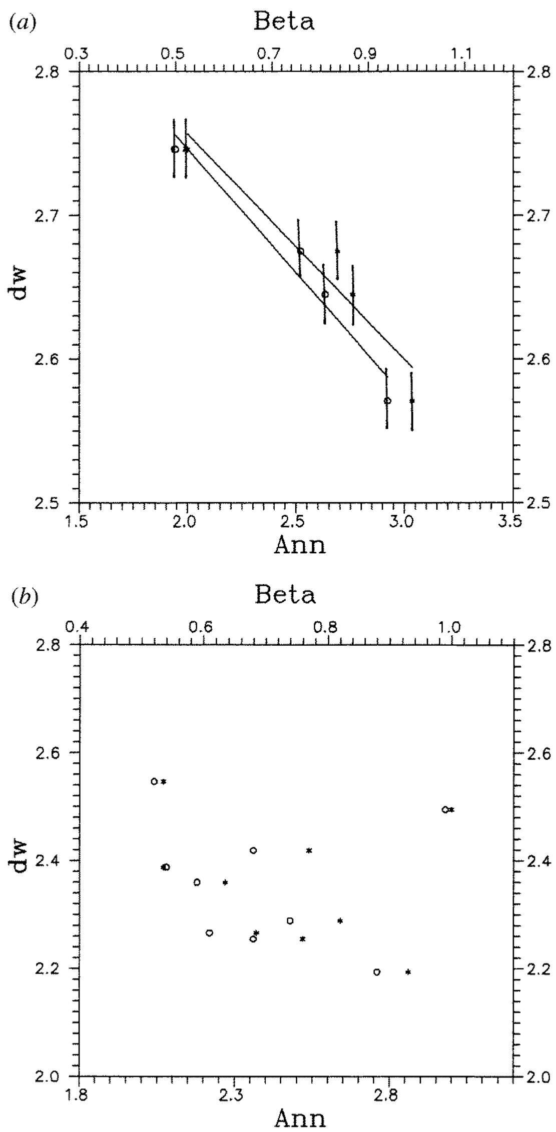

Figure 8. (*a*) d_w versus A_{nn} and β for the patterns of figure 3. (*b*) d_w versus A_{nn} and β for the patterns of figure 4.

is a wide variation in d_w ranging from 2.19 to 2.75. The difference arises only due to pattern geometry, since the d_f of all the patterns are the same. Table 1 also gives the values of β and A_{nn}.

It is seen that for the finitely ramified structures, there is a systematic almost linear variation in d_w with A_{nn} or, equivalently with β. An increase in A_{nn} or β represents an increase in compactness. The change in d_w with compactness can be understood from the following argument: when a walker diffuses through a very compact structure, it sees an almost Euclidean lattice for a considerable time, so d_w has a value closer to the Euclidean value two. A more fragmented pattern with a lower β gives a larger d_w, i.e. shows subdiffusive behaviour. This

Table 2. Results for $b = 4$ finitely ramified carpet patterns.

Carpet	Ram	A_{nn}	β	$d_w \pm 0.002$	$\delta \pm 0.01$
2(a)	f	2.00	0.50	2.538	0.01
2(b)	f	2.44	0.64	2.528	0.04
2(c)	f	2.67	0.75	2.524	0.06
2(d)	f	2.67	0.75	2.514	0.06

was also reported previously in [10]. The variation of d_w with β is plotted in figure 8(a). The $b = 4$ patterns which are finitely ramified show a similar behaviour. The results for these patterns are shown in table 2.

The infinitely ramified structures are much more complicated, a systematic variation in d_w with compactness is no longer observed. The points corresponding to infinitely ramified structures are also plotted in figure 8(b) to show that there is no apparent correlation between d_w and A_{nn} here.

Overall, we find $d_{w(max)}$ for our patterns, i.e. the highest obtained value is very close to $(1 + d_f)$, which is given by the relation for loopless and finitely ramified patterns, where $d_f = d_l$. The diffusion behaviour is strongly affected by d_l, but since we have no estimate of this exponent at present, we cannot check such a correlation.

Recently, a similar study on the diffusion behaviour of a family of Sierpinski carpets was performed by Kim *et al* [12]. They considered $b \times b$ squares with $l \times l$ squares blocked out. The patterns were symmetric along the X- and Y-directions. They gave theoretically estimated bounds for the values of d_w making certain approximations.

Let us see how our results fit in with these bounds. We calculate their bounds for $b = 7$ and $l = 5$ or $l = 6$, because in our case the number of blocked sites is 30 which is in between 5^2 and 6^2. The lower bound turns out to be 2.16 (for $b = 7$, $l = 5$), close to our result $d_{w(min)} = 2.19$. However, the upper bound with $b = 7$ and $l = 6$ is 2.32, whereas we find $d_{w(max)} = 2.75$. Possibly our values are higher because of the highly fragmented and loopless nature of a few of them. Another possibility is that their analysis does not hold for asymmetric patterns.

It appears that a considerable amount of work is still required to explain the diffusion behaviour in infinitely ramified fractals. The presence of loops and clusters appears to play an important role, this is discussed in the next section. Figure 8 shows a plot of d_w versus A_{nn}.

4.2. The estimation of δ

We now examine the deviation δ which measures the departure from the relation (6) for the different patterns.

A glance at table 1 shows that δ varies from a value close to zero to about 0.10. The error in estimation of δ is about 0.01, so it is not meaningful to attach too much importance to the exact values given. But it is quite clear that there is a deviation from the relation (6) for a number of patterns, which is outside the range of error. Let us try to see what characteristics of the pattern determine δ.

The $b = 4$ carpets do not show much variation in δ, so we take the $b = 7$ carpets first. If we group all the patterns into two classes, one with δ in the range 0–0.05 and the others with δ from 0.05–0.10, we cannot correlate the variation with β. Since the distribution of loops and clusters is not clear just by looking at the generator, let us look at the second-stage patterns. Some of them are shown in figures 5 and 6. In table 1 the finitely ramified patterns are denoted by ‘f’, and the infinitely ramified ones by ‘i’. It is quite evident that the finitely ramified patterns all have much higher values of δ than those in infinitely ramified patterns with the exception of pattern 4(i). However, the second stage of pattern 4(i) shown in figure 6(c) does not resemble the other infinitely ramified patterns, the second stage of one of which, 4(h), is shown in figure 6(b); but it is very similar to pattern 3(d), the second stage of which is shown in figure 6(a). In fact, most of its properties are very close to pattern 3(d). This suggests that whilst not apparent from the generator this pattern 4(i) should be grouped with the finitely ramified class as far its diffusion behaviour is concerned. That is, the random walker sees patterns 4(i) and 3(d) as very similar, although one of them is finitely ramified and the other is not. Exactly which parameter should be used to bring out the underlying similarity between these two is not very clear to us, but the clusters and loops in figures 3(d) and 4(i) can be seen to be very similar to the second-stage patterns in figures 6(a) and 6(c). A possible reason may be the presence of *unique* bottlenecks for paths spanning the carpet from left to right or top to bottom.

In general, the presence of loops reduces the value of δ as suggested by Dhar and Ramaswamy [8] and Nakanishi and Hermann [19], but in the case of finitely ramified patterns the presence of compact clusters appears to increase δ as reported earlier by Dasgupta *et al* [10].

In conclusion we have shown that (i) the diffusion exponent may vary widely for Sierpinski carpets with the same Hausdorff dimension, and (ii) the equivalence between d_s and d_n, the exponent for the number of distinct sites visited, does not always hold. Further work is necessary to pinpoint the features of the pattern responsible for determining this difference.

Acknowledgments

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Appendix. Implementation of the RW algorithm

We illustrate here how the algorithm for walks on infinite deterministic fractals works, with concrete examples.

Let us take the pattern with generator 2(d) and second stage 5(c), here $b = 4$.

The sites are identified by coordinates (x, y) starting with (0, 0) as the origin. Coordinates of the ten allowed sites are $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$ and $(3, 0)$ respectively.

Supposing the random walker chooses the point (72, 30) and we want to find out whether it is accessible or not.

The larger coordinate x lies between 4^3 and 4^4. So the point belongs to the stage $n = 4$. We consider the coarse-grained fourth stage pattern by looking at it through a grid of spacing $4^{n-1} = 64$. So our point belongs to the block $(1, 0)$ starting from the initial block $(0, 0)$, we call this the 'equivalent coordinate' of the fourth stage.

Comparing with the generator (figure 2(d)), this is seen to be accessible, so we move on to the third stage. Now the coordinates to be considered have reduced to the modulus of x with respect to 64 and the modulus of y with respect to 64, i.e. (8, 30). The coarse-grained grid is now $4^2 = 16$. Following the earlier procedure the equivalent coordinates for the third stage are $(0, 1)$, which is accessible in figure 2(d). In the second stage the modulus of 8 with respect to 16 and the modulus of 30 with respect to 16 gives $(8, 14)$, which when coarse grained on scale 4, give equivalent coordinates $(2, 3)$. This point is inaccessible according to figure 2(d), so we conclude that $(72, 30)$ is an inaccessible site.

In general, in the nth stage, the equivalent coordinates are given by: $x_r(\mathrm{eq})$ = integer part of (x_r/b^{r-1}); $y_r(\mathrm{eq})$ = integer part of (y_r/b^{r-1}). If (x_r, y_r) matches an allowed site, the coordinates carried over to the next stage are: $x_{r-1} = x_r \bmod b^{r-1}$; $y_{r-1} = y_r \bmod b^{r-1}$. If an equivalent coordinate of any stage does not match the list of accessible sites, the site under consideration is blocked, only those surviving up to stage 1 are accessible.

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